Fluid Simulation without Pressure

Wanho Choi
Seoul National University
Digital IDEa, Inc.

In-Yong Jeon
Seoul National University

Jong-Chul Yoon
FXGear, Inc.

Hyeong-Seok Ko
Seoul National University

Abstract

We suggests a new framework for enforcing the divergence-free condition on a velocity field. In this framework, the incompressibility is achieved by summing all the other nodal vorticities rather than via solving the Poisson equation. Our method is very simple to implement in both two and three dimensions and able to substitute for the conventional pressure projection step of the fluid simulation. In contrast with the pressure projection step, the proposed method can be efficiently parallelizable on multi-core or GPU architectures.

1 Introduction

Numerical simulations of inviscid and incompressible flows have two main parts: the advection step and the pressure projection step. Parallelizing the advection step is relatively easy when the simulator is based on the semi-Lagrangian advection scheme. However, parallelizing the pressure projection step is not easy because one must solve the Poisson equation, which is globally coupled system. On the other hand, we solve the *N*-vortex problem by modifying the governing equations and using the *Biot-Savart law*. As a result, our method is simpler and more efficient for the parallelization than the conventional pressure prejection method. It is also not required any iteratons.

2 Our Approach

Usually, incompressible inviscid flows are simulated according to the following governing equations,

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \mathbf{f} \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

where t is time, $\mathbf{u}$ is the velocity, p is the pressure, and $\mathbf{f}$ is the external force. In combination with the pressure term of equation (1), equation (2) reduces to the Poisson equation, $\nabla\cdot\nabla p = \nabla\cdot\widetilde{\mathbf{u}}$. Here, $\widetilde{\mathbf{u}}$ is the intermediate velocity. After solving p from the Poisson equation, the divergence-free velocity field $\mathbf{u}^{\text{new}}$ for the next time step can be obtained according to $\mathbf{u}^{\text{new}} = \widetilde{\mathbf{u}} - \nabla p$.

On the other hand, we note that the intermediate velocity $\widetilde{\mathbf{u}}$ can be decomposed into the sum of conservative and solenoidal parts (the *Helmholtz-Hodge* decomposition),

$$\widetilde{\mathbf{u}} = \widetilde{\mathbf{u}}_{\text{conservative}} + \widetilde{\mathbf{u}}_{\text{solenoidal}} \tag{3}$$

such that $\nabla\times\widetilde{\mathbf{u}}_{\text{conservative}} = 0$ and $\nabla\cdot\widetilde{\mathbf{u}}_{\text{solenoidal}} = 0$. Hence, the solenoidal velocity field $\widetilde{\mathbf{u}}_{\text{solenoidal}}$ is identical to the divergence-free velocity field $\mathbf{u}^{\text{new}}$ to be determined for the next time step.

Taking the curl on both sides of equation (3) gives $\nabla\times\mathbf{u}^{\text{new}} = \nabla\times\widetilde{\mathbf{u}}$. Therefore, the velocity field $\mathbf{u}^{\text{new}}$ can be calculated according to the *Biot-Savart law* [Park and Kim 2005]:

$$\mathbf{u}^{\text{new}}(\mathbf{p}) = \frac{1}{A\pi}\int_{\mathbf{x}} \frac{\widetilde{\boldsymbol{\omega}}(\mathbf{x})\times(\mathbf{p}-\mathbf{x})}{\|\mathbf{p}-\mathbf{x}\|^d}\,d\mathbf{x} \tag{4}$$

Figure 1: 2D real-time fluid simulation using CUDA. A 512×512 regular grid with 64×64 clusters was used.

where $\widetilde{\boldsymbol{\omega}} = \nabla\times\widetilde{\mathbf{u}}$ and $A=2$, $d=2$ for two dimensions(2D), and $A=4$, $d=3$ for three dimensions(3D). Equation (4) can be transformed into a discrete form:

$$\mathbf{u}^{\text{new}}(\mathbf{p}) = \frac{h^d}{A\pi}\sum_{i\in\text{nodes}} \frac{\widetilde{\boldsymbol{\omega}}_i\times(\mathbf{p}-\mathbf{x}_i)}{\|\mathbf{p}-\mathbf{x}_i\|^d} \tag{5}$$

where h is the cell size, $\mathbf{x}_i$ and $\widetilde{\boldsymbol{\omega}}_i$ are the position and vorticity of the i-th node, and $\mathbf{p}$ is the position at which we would like to calculate the velocity. Now, it reduces to the *N*-vortex problem. A general solution to this problem is the Fast Multipole Method (FMM), [Greengard and Rokhlin 1987], which provides an approximate solution while reducing the computation time by clustering long-range forces using truncated series expansions. However, implementation of FMM is particularly complicated in 3D.

We suggest a new method, Fast Vorticity Summation Method (FVSM). We apply the heuristic assumption that if a cluster is sufficiently far from the position $\mathbf{p}$, $\|\mathbf{p}-\mathbf{x}_i\|$ in the denominator of equation (5) can be approximated by $\|\mathbf{p}-\mathbf{x}_{\text{ref}}\|$, where $\mathbf{x}_{\text{ref}}$ is the reference point of the cluster. With the above assumption, the velocity at $\mathbf{p}$ can be approximated by

$$\mathbf{u}^{\text{new}}(\mathbf{p}) \approx \frac{h^d}{A\pi}\frac{1}{\|\mathbf{p}-\mathbf{x}_{\text{ref}}\|^d}\left\{\left(\sum\widetilde{\boldsymbol{\omega}}_i\right)\times\mathbf{p} - \left(\sum\widetilde{\boldsymbol{\omega}}_i\times\mathbf{x}_i\right)\right\} \tag{6}$$

where i is the nodal index within the cluster. We set the $\mathbf{x}_{\text{ref}}$ as the weighed average point by the magnitude of the vorticity. As one can see, for sufficiently distant clusters, only three vectors($\sum\widetilde{\boldsymbol{\omega}}_i$, $\sum\widetilde{\boldsymbol{\omega}}_i\times\mathbf{x}_i$ and $\mathbf{x}_{\text{ref}}$) have to be transmitted. Before proceeding to the main calculation, these three vectors are calculated for each cluster independently in advance. Neighboring clusters should share their nodal velocities for the calculation of vorticities and the direct summation, which is the same procedure as is used in the FMM. However, these data are transmitted only between neighboring clusters, and the transmission is also necessary for the semi-Lagrangian advection step.

References

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