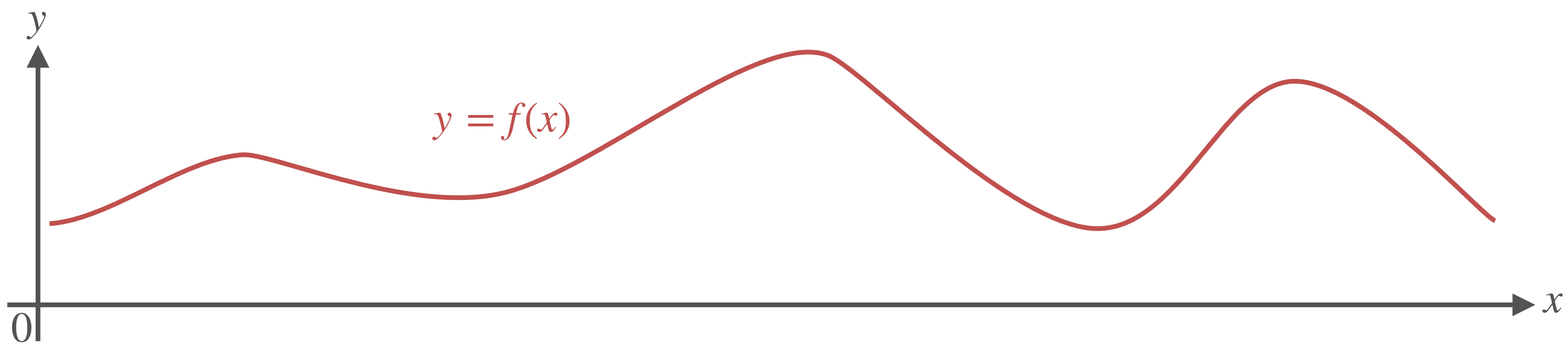
PDE to Ax=b

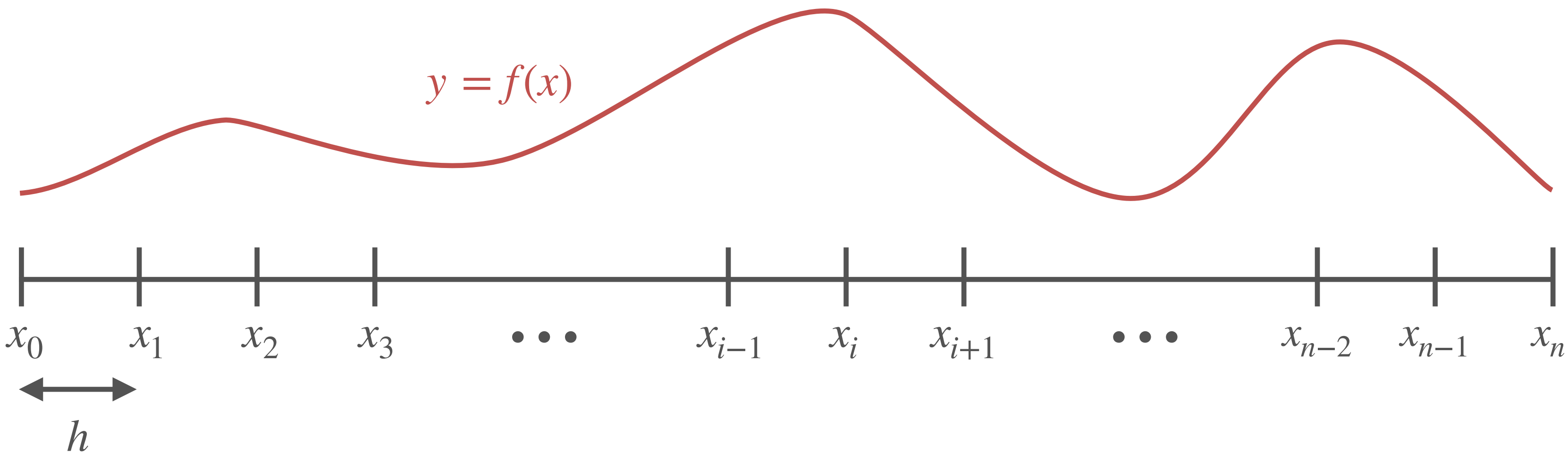
Dexter Studios R&D

Wanho Choi

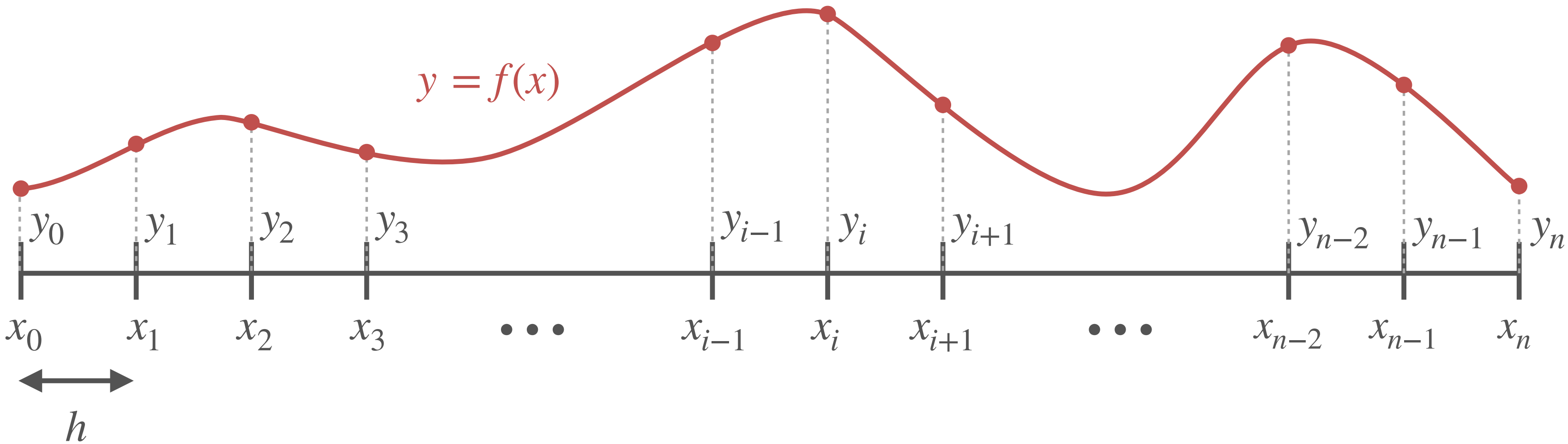
A Given 1D Function: y=f(x)



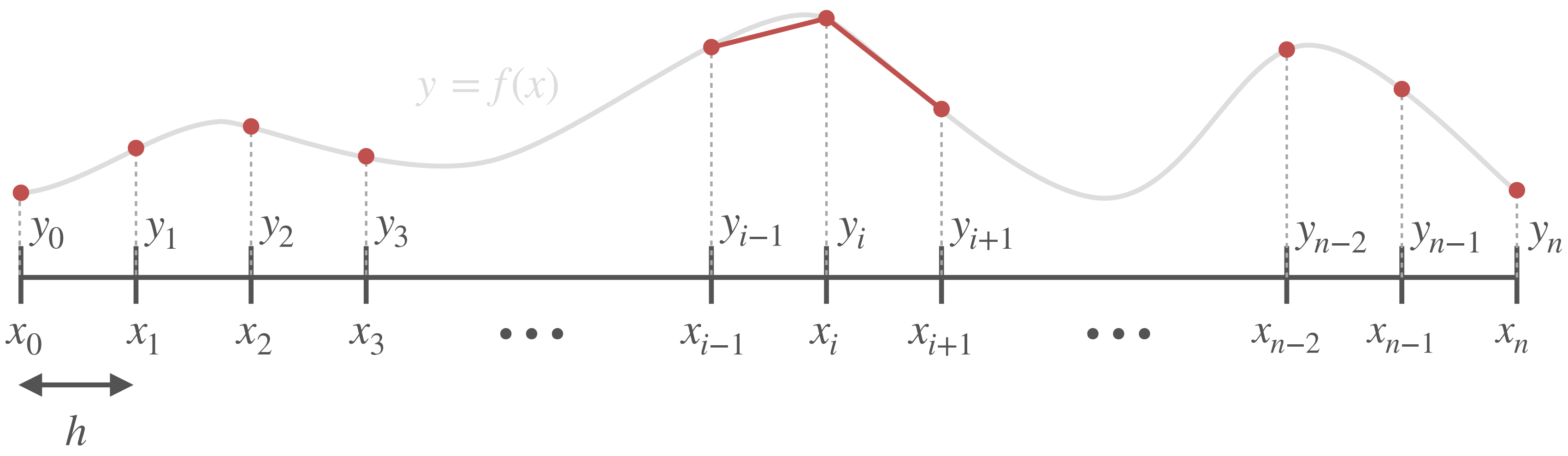
Sampling using a 1D Grid



Sampling using a 1D Grid



Finite Differencing for the i-th Point



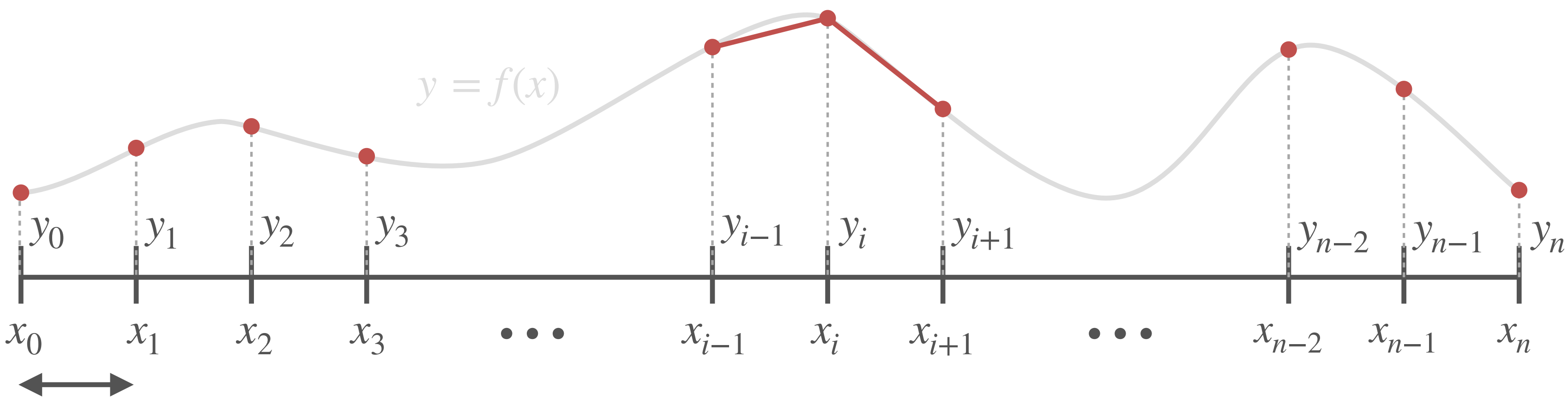
backward differentiation $$y'(x_i) = \frac{y(x_{i+1}) - y(x_i)}{h}$$

forward differentiation $$y'(x_i) = \frac{y(x_i) - y(x_{i-1})}{h}$$

central differentiation $$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h}$$

If $h \to 0$, *they all converge to the same value .*

The 2nd Derivative for the i-th Point

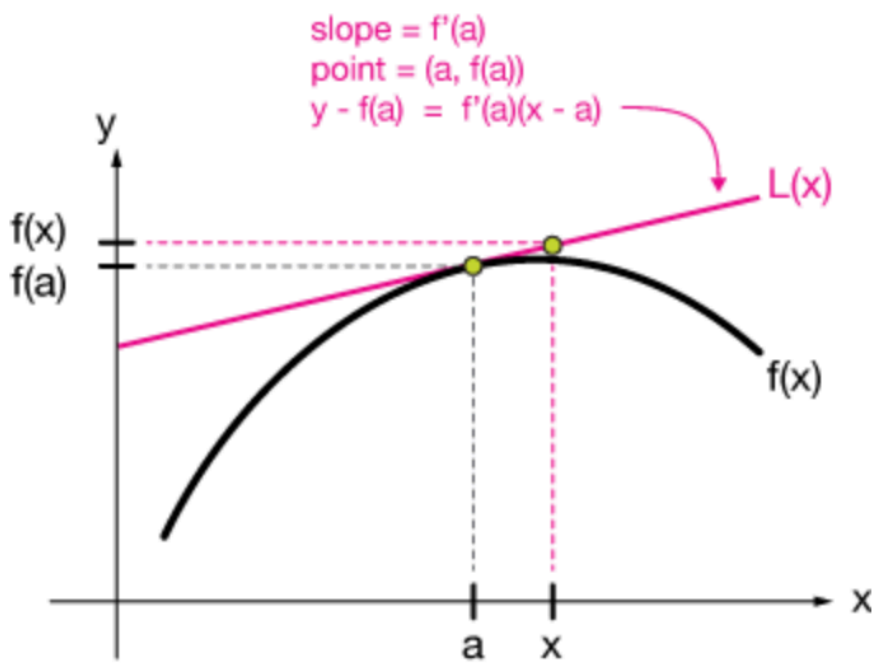


$$y''(x_i) = \frac{y'(x_{i+0.5}) - y'(x_{i-0.5})}{h} = \frac{\dfrac{y(x_{i+1}) - y(x_i)}{h} - \dfrac{y(x_i) - y(x_{i-1})}{h}}{h}$$

$$\therefore y'' = \frac{y(x_{i-1}) - 2y(x_i) + y(x_{i+1})}{h^2}$$

Taylor Expansion Series

$$y(x_i + h) = y(x_i) + \frac{y'(x_i)h}{1!} + \frac{y''(x_i)h^2}{2!} + \frac{y'''(x_i)h^3}{3!} + \cdots$$



http://xaktly.com/TaylorSeries.html

$$f(x) \approx f(a) + f'(x)(x - a).$$

Taylor Expansion Series

$$y(x_i + h) = y(x_i) + \frac{y'(x_i)h}{1!} + \frac{y''(x_i)h^2}{2!} + \frac{y'''(x_i)h^3}{3!} + \cdots$$

$$y(x_i - h) = y(x_i) - \frac{y'(x_i)h}{1!} + \frac{y''(x_i)h^2}{2!} - \frac{y'''(x_i)h^3}{3!} + \cdots$$

Taylor Expansion Series

$$y(x_i + h) = y(x_i) + \frac{y'(x_i)h}{1!} + \frac{y''(x_i)h^2}{2!} + \frac{y'''(x_i)h^3}{3!} + \cdots \qquad \text{(a)}$$

$$y(x_i - h) = y(x_i) - \frac{y'(x_i)h}{1!} + \frac{y''(x_i)h^2}{2!} - \frac{y'''(x_i)h^3}{3!} + \cdots \qquad \text{(b)}$$

(a)-(b): $$y'(x_i) = \frac{y(x_i + h) - y(x_i - h)}{2h} + O(h^3)$$

(a)+(b): $$y''(x_i) = \frac{y(x_i - h) - 2y(x_i) + y(x_i - h)}{2h} + O(h^4)$$

A Simple PDE Problem

$$y'' = f(x) \quad on \quad [0,1] \quad with \quad y(0) = y(1) = 0$$

➡ $y(x_i) = ? \qquad (i = 0,1,2,\cdots,n)$

Finite Differencing

$$y'' = f(x) \quad on \quad [0,1] \quad with \quad y(0) = y(1) = 0$$

➡ $$y(x_i) = ? \qquad (i = 0,1,2,\cdots,n)$$

$$\frac{y(x_{i-1}) - 2y(x_i) + y(x_{i+1})}{h^2} \approx y''(x_i) = f_i$$

Finite Differencing

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➡ $y(x_i) = ? \qquad (i = 0,1,2,\cdots,n)$

$$\frac{y(x_{i-1}) - 2y(x_i) + y(x_{i+1})}{h^2} \approx y''(x_i) = f_i$$

$$\therefore y(x_{i-1}) - 2y(x_i) + y(x_{i+1}) \approx h^2 f_i$$

For Every Point

$$y'' = f(x) \;\; on \;\; [0,1] \;\; with \;\; y(0) = y(1) = 0$$

$$\Rightarrow \quad y(x_i) = ? \qquad (i = 0,1,2,\cdots,n)$$

$$y_{-1} - 2y_0 + y_1 = h^2 f_0$$

$$y_0 - 2y_1 + y_2 = h^2 f_1$$

$$y_1 - 2y_2 + y_3 = h^2 f_2$$

$$\ddots$$

$$y_{n-2} - 2y_{n-1} + y_n = h^2 f_n$$

Linear System of Matrix Equation

$$y'' = f(x) \;\; on \;\; [0,1] \;\; with \;\; y(0) = y(1) = 0$$

$$\Rightarrow \quad y(x_i) = ? \qquad (i = 0,1,2,\cdots,n)$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} h^2 f_0 \\ h^2 f_1 \\ h^2 f_2 \\ h^2 f_3 \\ \vdots \\ h^2 f_{n-3} \\ h^2 f_{n-2} \\ h^2 f_{n-1} \\ h^2 f_n \end{bmatrix}$$

Linear System of Matrix Equation

$$y'' = f(x) \;\; on \;\; [0,1] \;\; with \;\; y(0) = y(1) = 0$$

$$\Rightarrow \quad y(x_i) = \, ? \qquad (i = 0,1,2,\cdots,n)$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} h^2 f_0 \\ h^2 f_1 \\ h^2 f_2 \\ h^2 f_3 \\ \vdots \\ h^2 f_{n-3} \\ h^2 f_{n-2} \\ h^2 f_{n-1} \\ h^2 f_n \end{bmatrix}$$

$$\mathbf{Ax} = \mathbf{b}$$

Q & A