Gram-Schmidt Process

Dexter Studios R&D

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Linear Combination of Vectors

- Combining m n-dimensional vectors
  using scalar multiplications and vector additions

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_m\mathbf{x}_m = \sum_{i=1}^{m} a_i\mathbf{x}_i$$

$$\left(a_i \in \mathbb{R}, \quad \mathbf{x}_i \in \mathbb{R}^n, \quad \mathrm{m} \in \mathbb{N}\right)$$

Vector Space

- A n-dimensional space in which any linear combination of a set of vectors is also in that space

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_m\mathbf{x}_m = \sum_{i=1}^{m} a_i\mathbf{x}_i$$

$$\left(a_i \in \mathbb{R}, \quad \mathbf{x}_i \in \mathbb{R}^n, \quad \mathrm{m} \in \mathbb{N}\right)$$

$$\mathbf{y} \in \mathbb{R}^n$$

Linearly Dependent / Independent

- $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_m$: linearly dependent
  iff $a_1, a_2, a_3, \cdots, a_m$ exist for

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_m\mathbf{x}_m = \sum_{i=1}^{m} a_i\mathbf{x}_i = \mathbf{0}$$

$$\left(a_1 \times a_2 \times a_3 \times \cdots \times a_m \neq 0\right)$$

- Otherwise, they are linearly independent.

Basis

- A set of n vectors in a n-dimensional space, which are linearly independent and every vector in that space can be reproduced by a linear combination of this set.

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + \cdots + a_n\mathbf{e}_n \in \mathbb{R}^3$$

$$\left(a_i \in \mathbb{R}, \quad \mathbf{e}_i \in \mathbb{R}^n\right)$$

Orthonormal Basis

- A set of n vectors in a n-dimensional space, which are orthonormal each other and every vector in that space can be reproduced by a linear combination of this set.

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + \cdots + a_n\mathbf{e}_n \in \mathbb{R}^3$$

$$\left(a_i \in \mathbb{R}, \quad \mathbf{e}_i \in \mathbb{R}^n\right)$$

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

Span

- All linear combination of linearly independent vectors

$$span(\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + \cdots + a_n\mathbf{e}_n$$

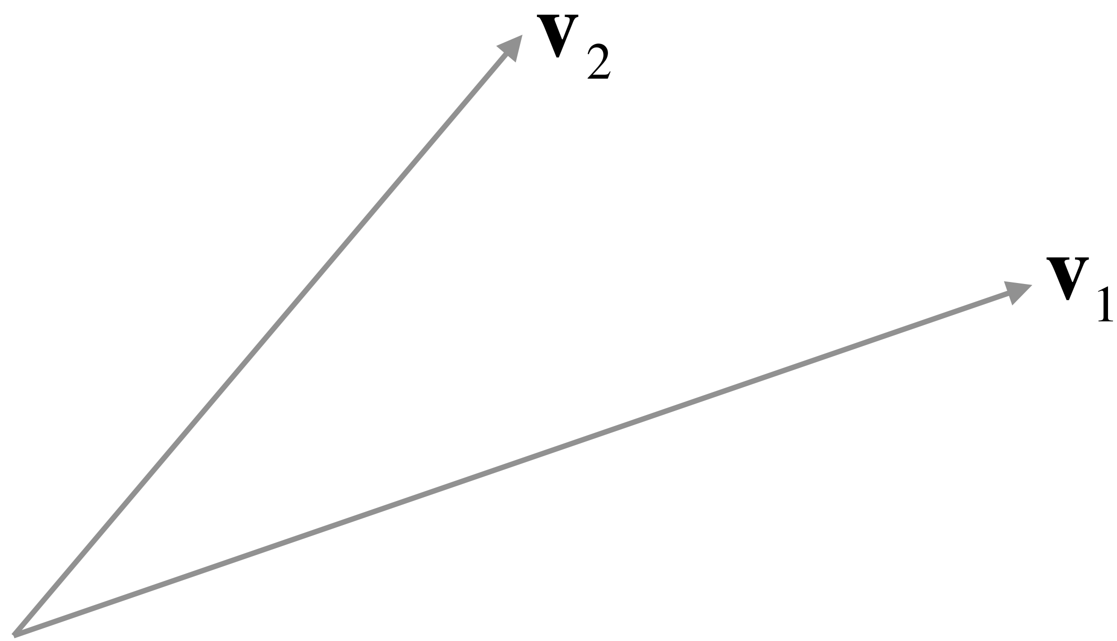
$$\left(a_i \in \mathbb{R}, \quad \mathbf{e}_i \in \mathbb{R}^n\right)$$

Gram-Schmidt Process

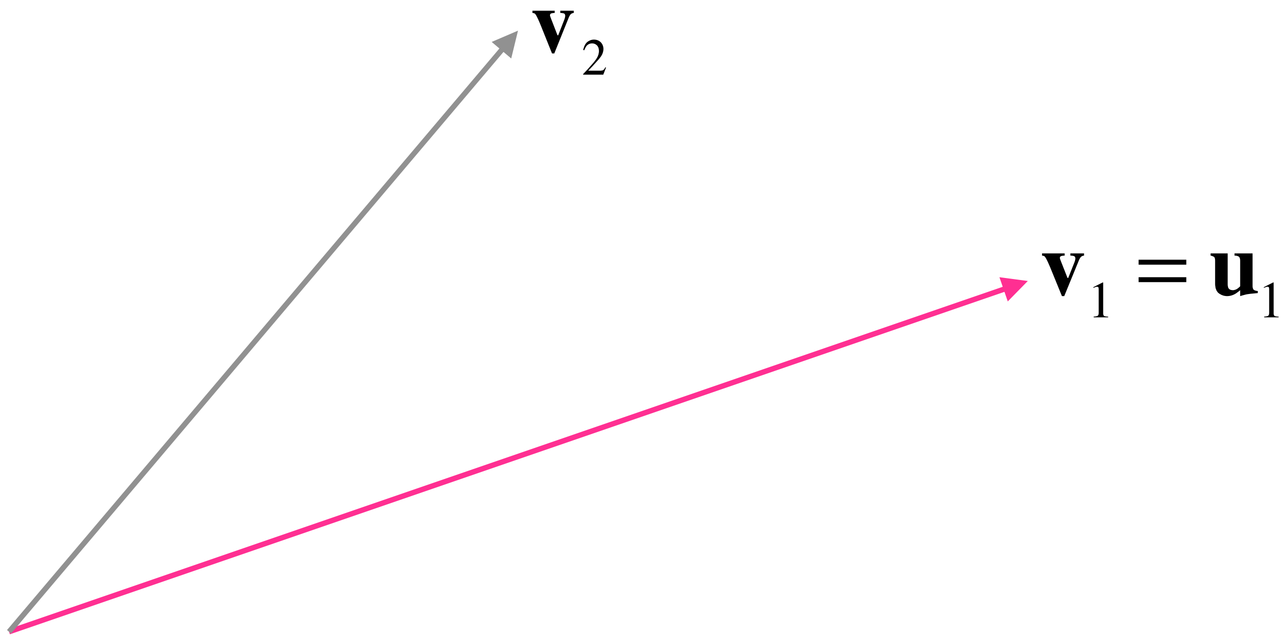
- Input: $\{\mathbf{v}_i\}$
  Non-orthogonal set of independent vectors

- Output: $\{\mathbf{u}_i\}$
  Orthogonal set of vectors

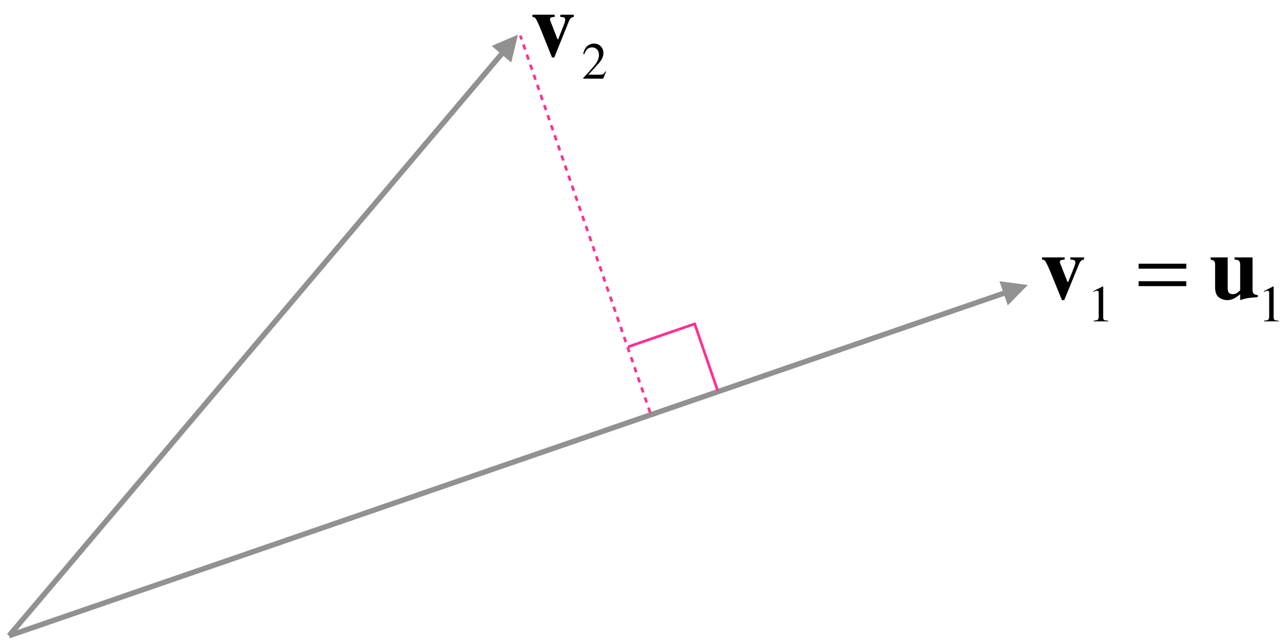
Gram-Schmidt Process



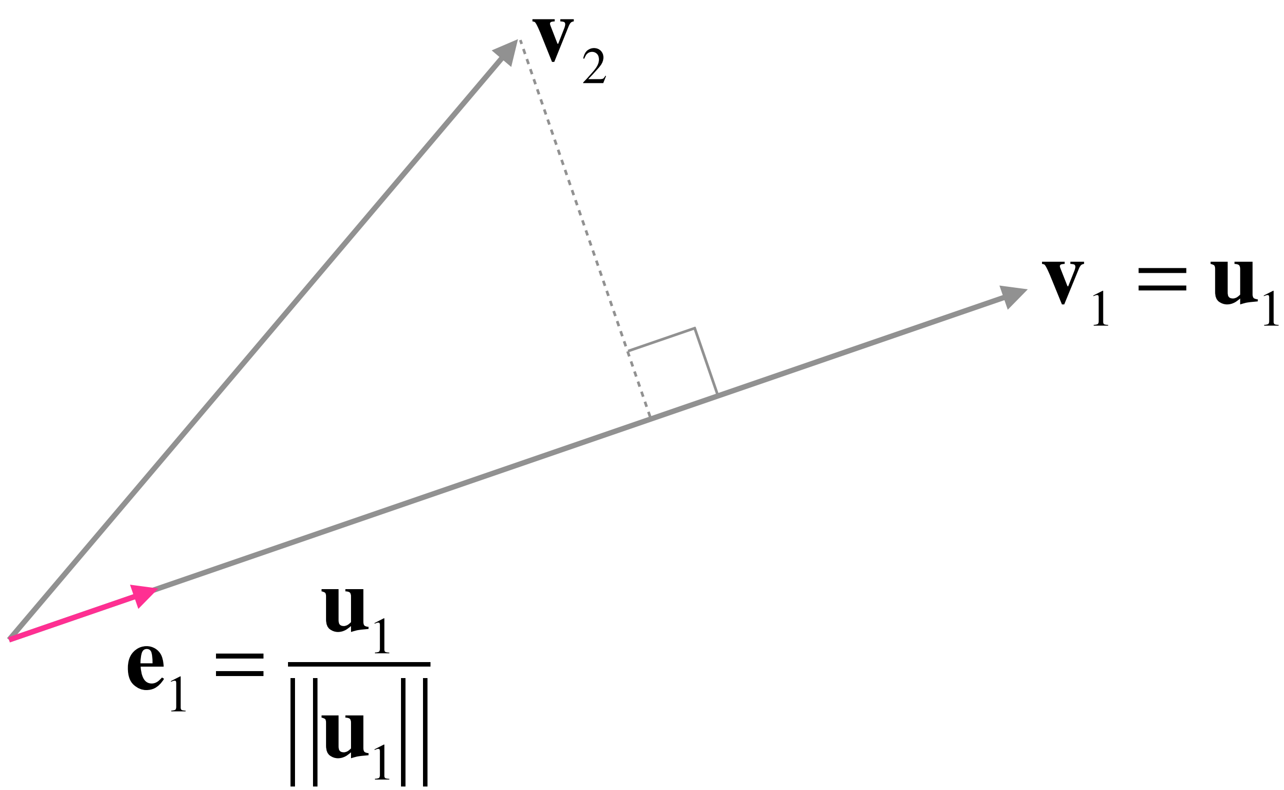
Gram-Schmidt Process



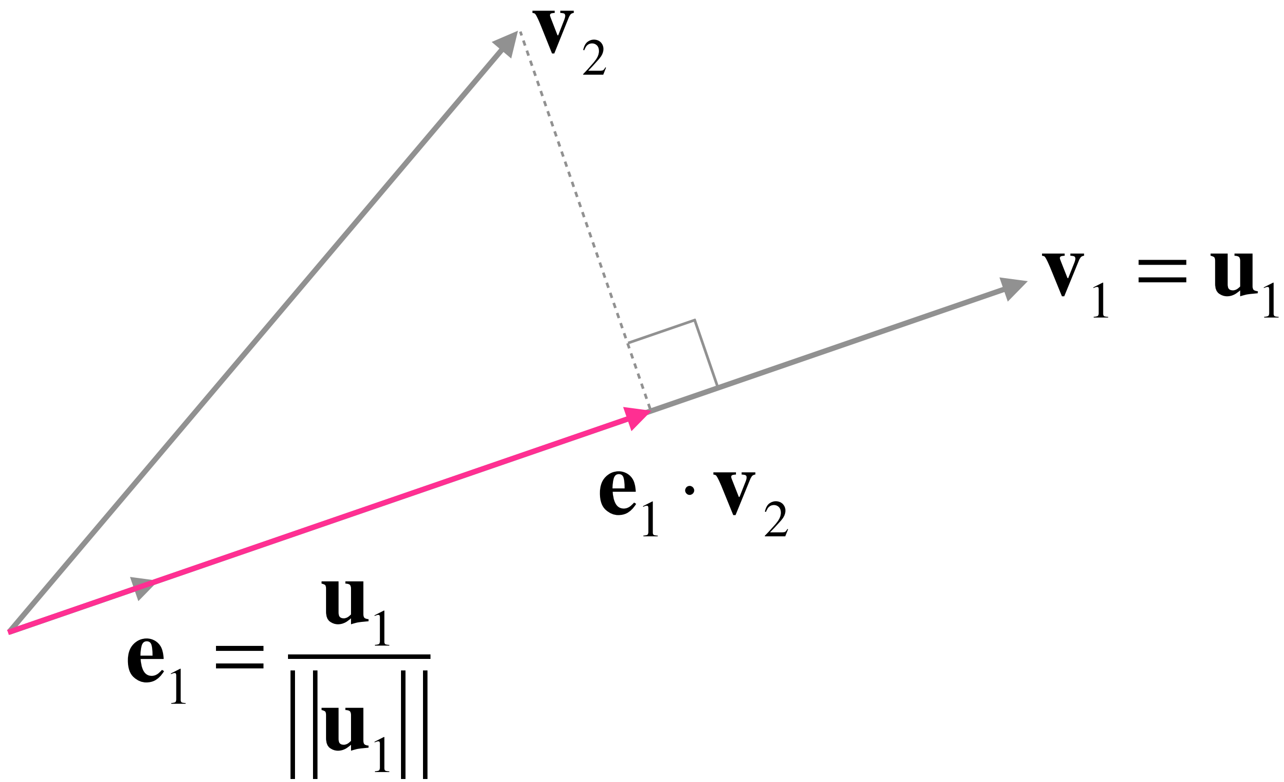
Gram-Schmidt Process



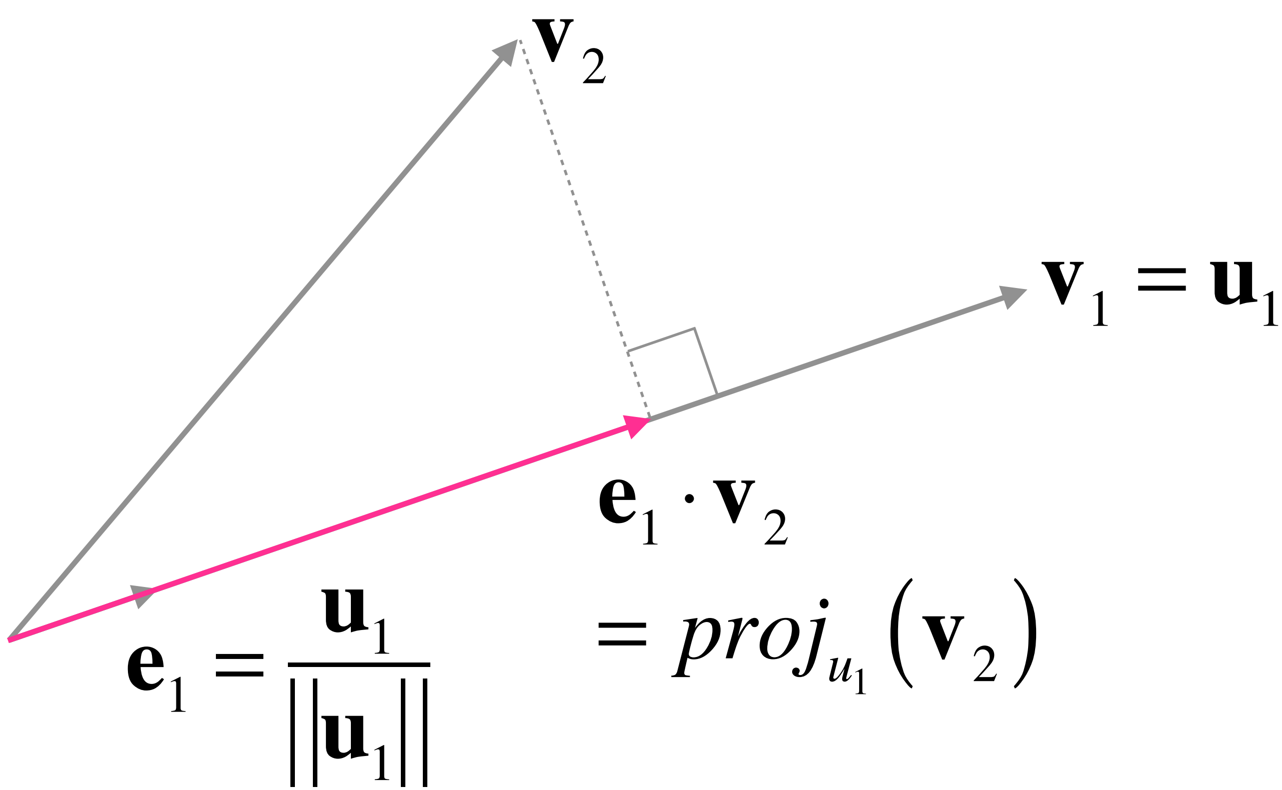
Gram-Schmidt Process



Gram-Schmidt Process



Gram-Schmidt Process



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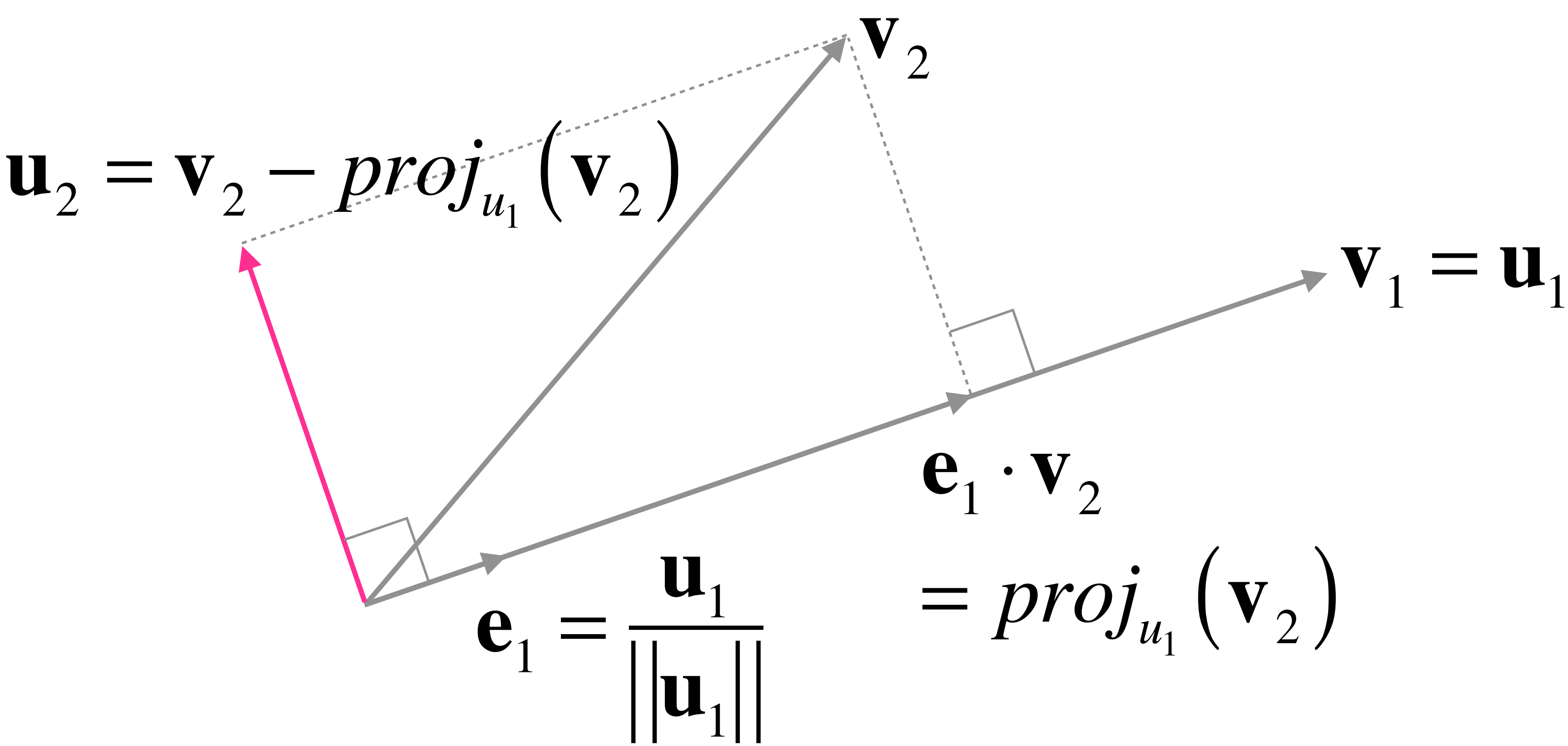

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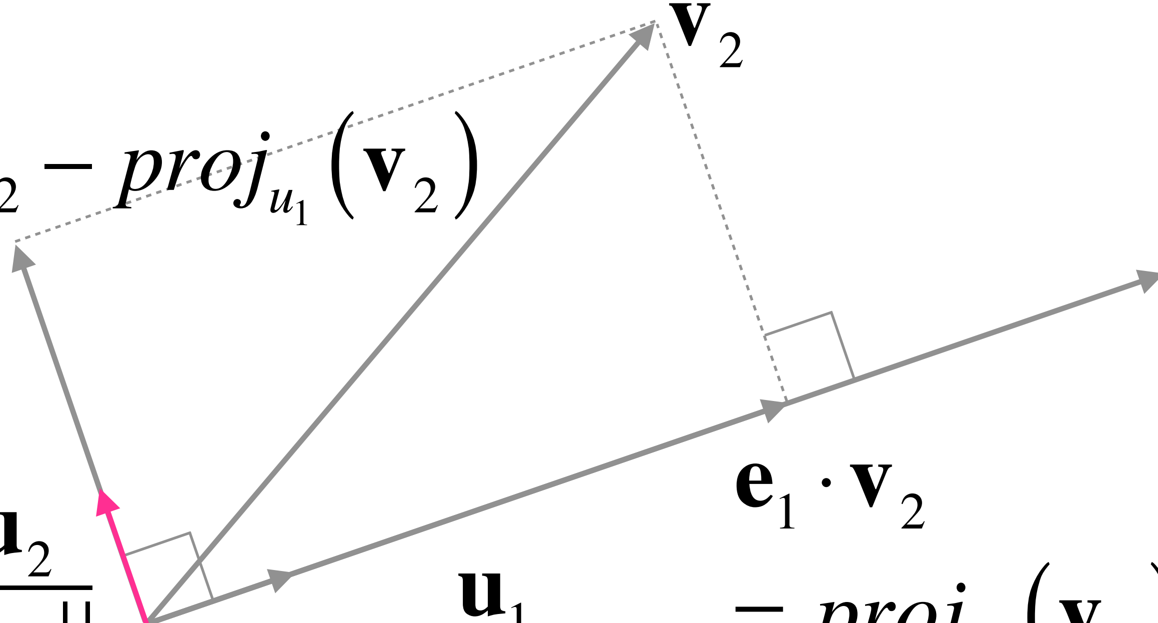

Gram-Schmidt Process

$$\mathbf{u}_1 = \mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{v}_2 - proj_{u_1}\left(\mathbf{v}_2\right)$$

$$\mathbf{u}_3 = \mathbf{v}_3 - proj_{u_1}\left(\mathbf{v}_3\right) - proj_{u_2}\left(\mathbf{v}_3\right)$$

$$\vdots$$

$$\mathbf{u}_k = \mathbf{v}_k - \sum_{j=1}^{k} proj_{u_j}\left(\mathbf{v}_k\right)$$

Gram-Schmidt Process

$$\mathbf{u}_1 = \mathbf{v}_1 \qquad \mathbf{e}_1 = \mathbf{u}_1 / \|\mathbf{u}_1\|$$

$$\mathbf{u}_2 = \mathbf{v}_2 - proj_{u_1}(\mathbf{v}_2) \qquad \mathbf{e}_2 = \mathbf{u}_2 / \|\mathbf{u}_2\|$$

$$\mathbf{u}_3 = \mathbf{v}_3 - proj_{u_1}(\mathbf{v}_3) - proj_{u_2}(\mathbf{v}_3) \qquad \mathbf{e}_3 = \mathbf{u}_3 / \|\mathbf{u}_3\|$$

$$\vdots \qquad \vdots$$

$$\mathbf{u}_k = \mathbf{v}_k - \sum_{j=1}^{k} proj_{u_j}(\mathbf{v}_k) \qquad \mathbf{e}_k = \mathbf{u}_k / \|\mathbf{u}_k\|$$

Gram-Schmidt Process

$$\mathbf{u}_1 = \mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{v}_2 - proj_{u_1}(\mathbf{v}_2)$$

$$\mathbf{u}_3 = \mathbf{v}_3 - proj_{u_1}(\mathbf{v}_3) - proj_{u_2}(\mathbf{v}_3)$$

$$\vdots$$

$$\mathbf{u}_k = \mathbf{v}_k - \sum_{j=1}^{k} proj_{u_j}(\mathbf{v}_k)$$

$$\boxed{\begin{array}{l} \mathbf{e}_1 = \mathbf{u}_1 / \|\mathbf{u}_1\| \\ \mathbf{e}_2 = \mathbf{u}_2 / \|\mathbf{u}_2\| \\ \mathbf{e}_3 = \mathbf{u}_3 / \|\mathbf{u}_3\| \\ \vdots \\ \mathbf{e}_k = \mathbf{u}_k / \|\mathbf{u}_k\| \end{array}}$$

orthonormal basis

Q & A