Barycentric Coordinates

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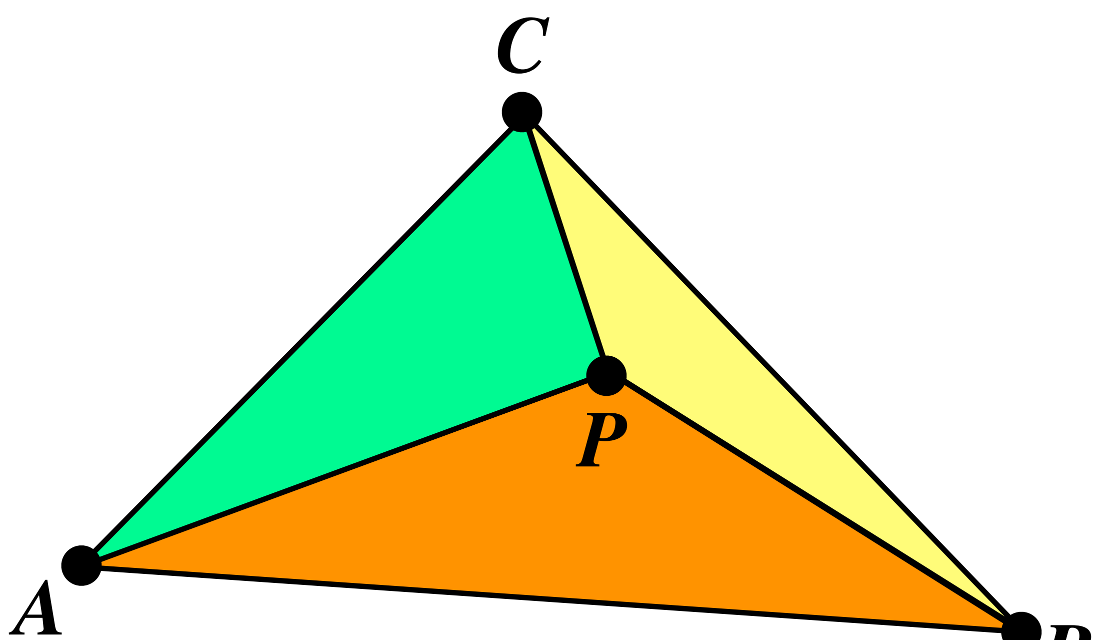
Barycentric Coordinates

- **A coordinate system in which the location of a point of a simplex**

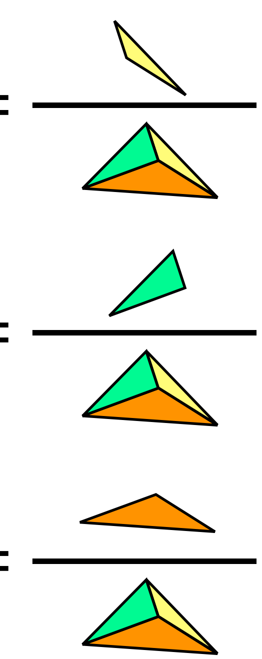


$$P = w_A \times A + w_B \times B + w_C \times C$$

$$w_A = \frac{\Delta PBC}{\Delta ABC} =$$

$$w_B = \frac{\Delta PCA}{\Delta ABC} =$$

$$w_C = \frac{\Delta PAB}{\Delta ABC} =$$

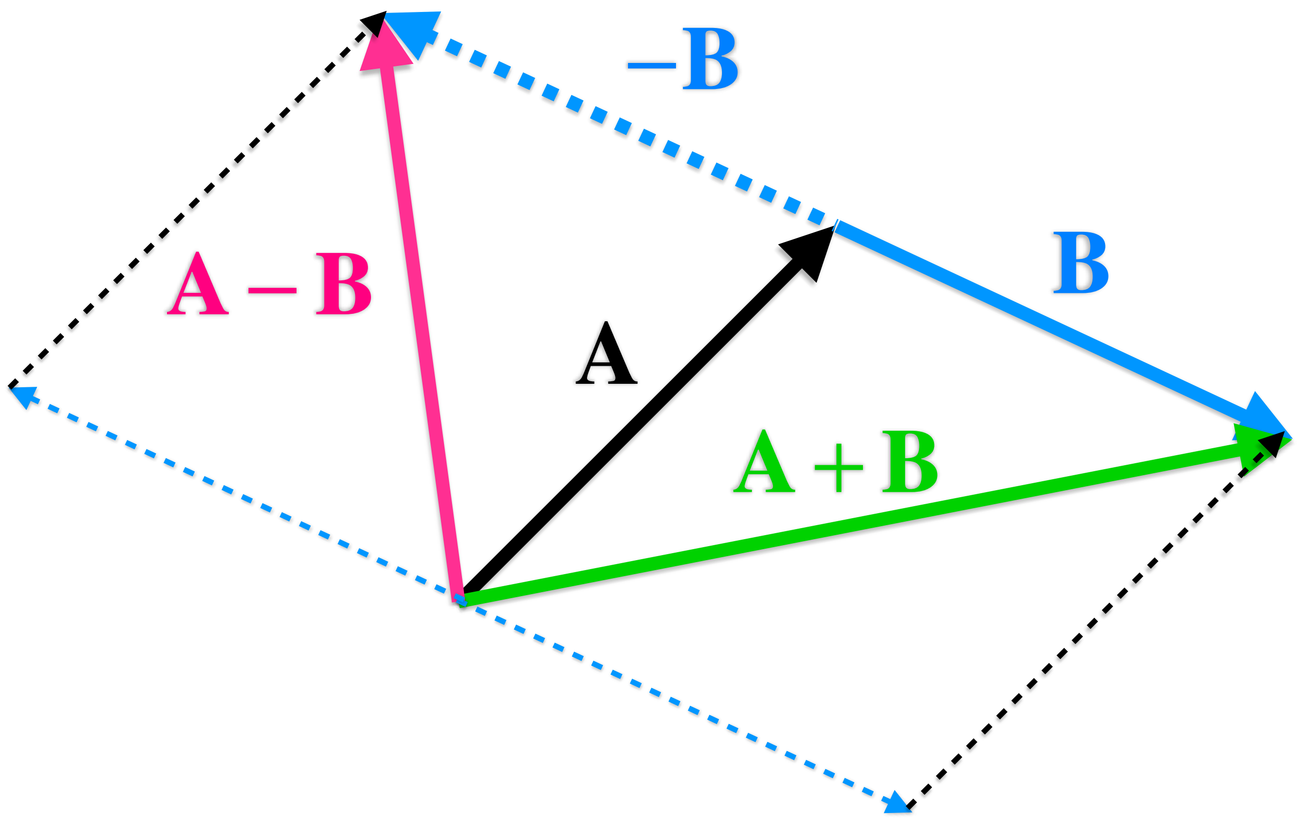


inside condition

$$0 \leq w_A, w_B, w_C \leq 1 \qquad w_A + w_B + w_C = 1$$

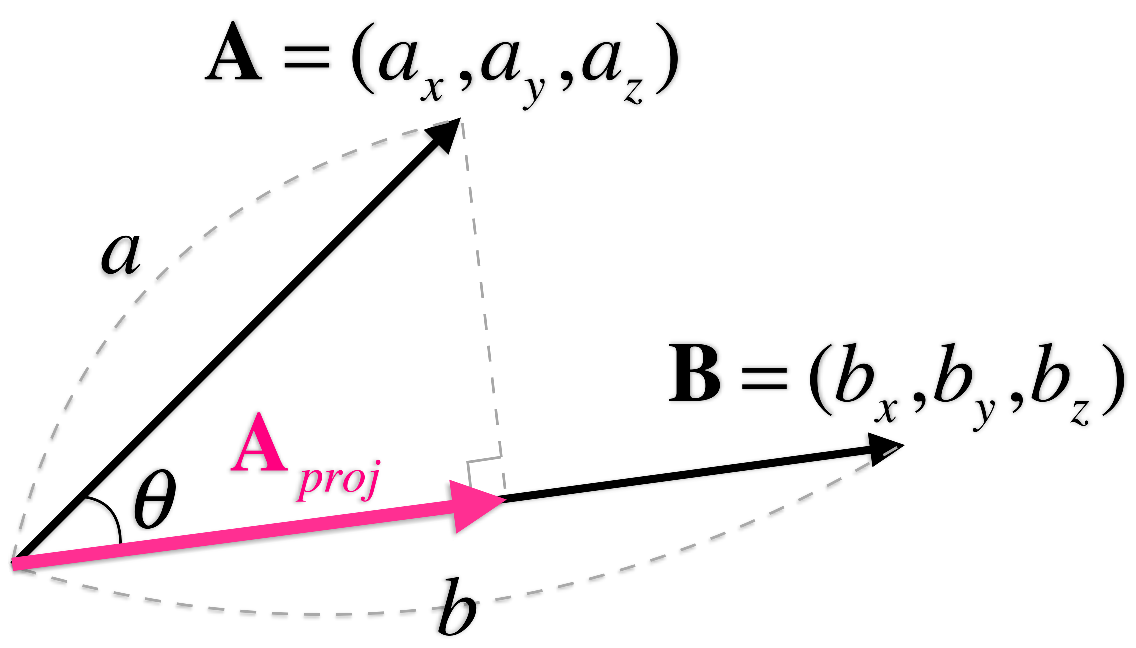
Vector + & -

- **Component-wise operation**



Dot Product

- **Projection**

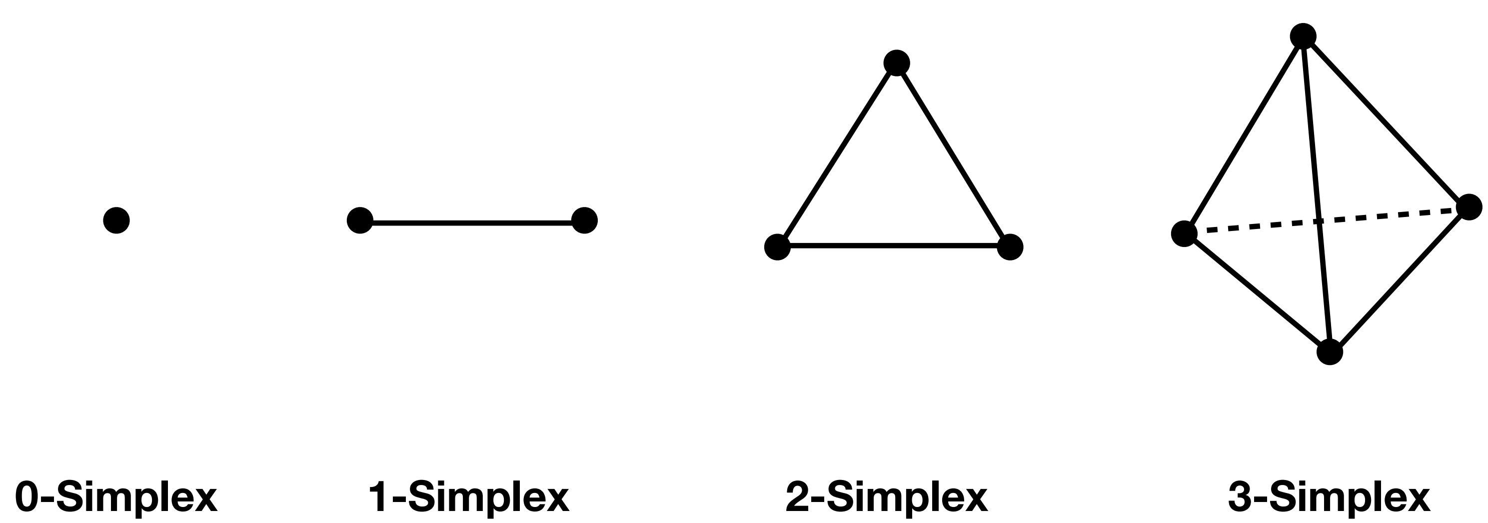


$$\mathbf{A} \cdot \mathbf{B} \equiv a_x b_x + a_y b_y + a_z b_z$$

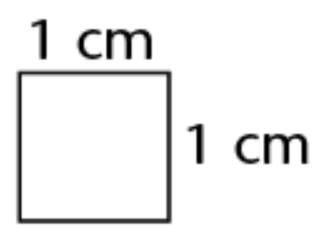
$$= ab \cos\theta$$

Simplex

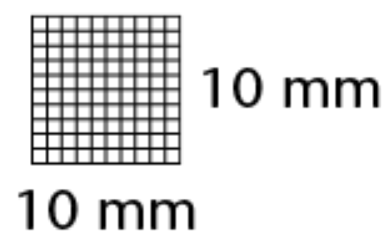
- **A generalization of the notion of a triangle or tetrahedron to arbitrary dimensions**



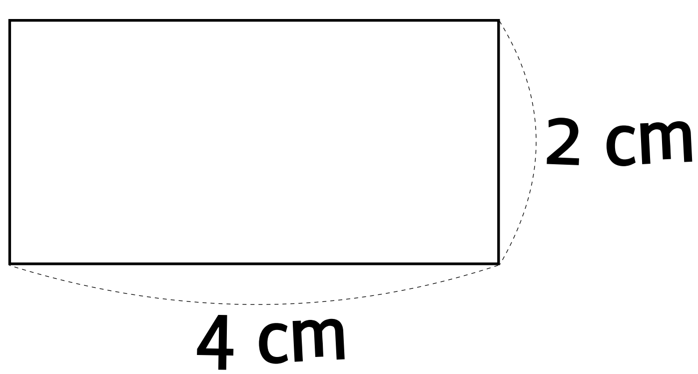
The Area



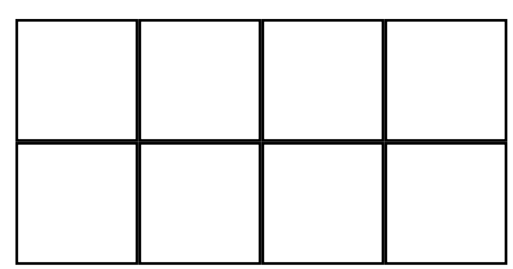
promise $\equiv 1\ cm^2$



$= 100\ mm^2$

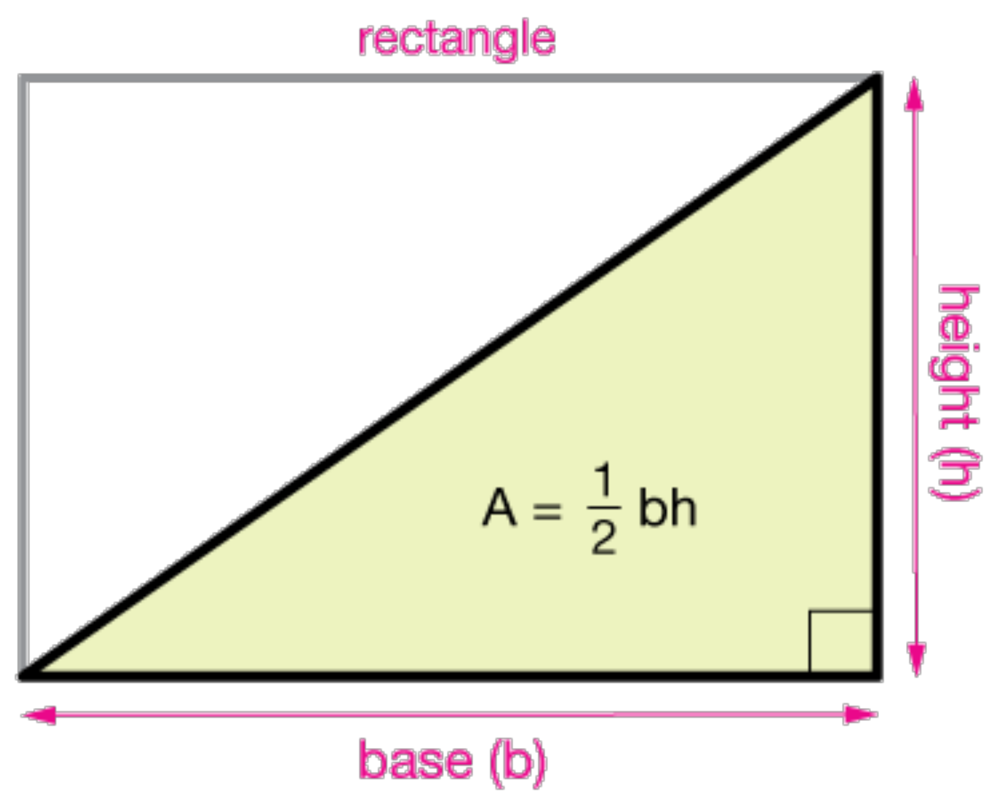


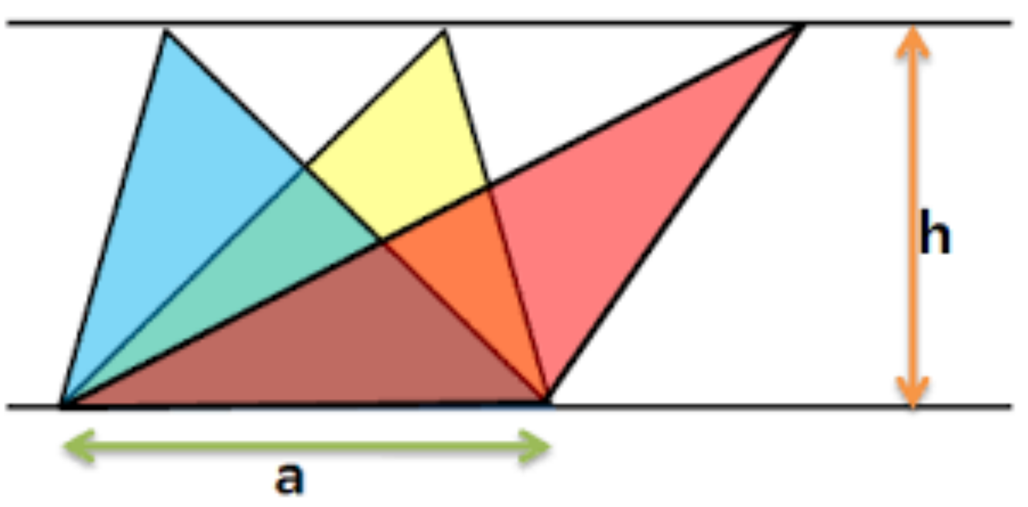
$=$



$= 1\ cm^2 \times 8\ EA = 8\ cm^2$

The Area of a Triangle





2x2 Matrix Inverse

System of equations:

$$x + 2y = 5$$
$$3x + 5y = 14$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

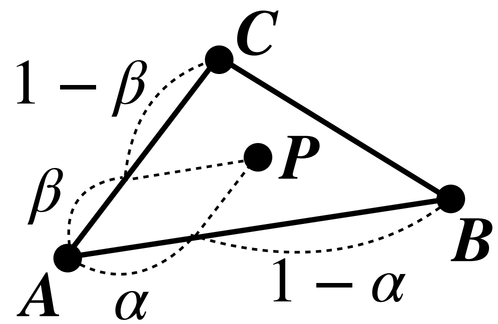
$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \end{bmatrix} \longrightarrow AX = B \longrightarrow X = A^{-1}B$$

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}^{-1} = \frac{1}{1(5) - 2(3)}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = -1\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

https://study.com/academy/lesson/using-an-inverse-matrix-to-solve-a-system-of-linear-equations.html



$$\boldsymbol{AB} = \boldsymbol{B} - \boldsymbol{A}$$
$$\boldsymbol{AC} = \boldsymbol{C} - \boldsymbol{A}$$
$$\boldsymbol{Q} = \boldsymbol{P} - \boldsymbol{A}$$

$$\alpha \boldsymbol{AB} + \beta \boldsymbol{AC} + \boldsymbol{A} = \boldsymbol{P}$$

$$\Longleftrightarrow \alpha \boldsymbol{AB} + \beta \boldsymbol{AC} = \boldsymbol{P} - \boldsymbol{A}$$

$$\Longleftrightarrow \alpha \boldsymbol{AB} + \beta \boldsymbol{AC} = \boldsymbol{Q}$$

$$\Longleftrightarrow \begin{matrix} \alpha \boldsymbol{AB}.x + \beta \boldsymbol{AC}.x = \boldsymbol{Q}.x \\ \alpha \boldsymbol{AB}.y + \beta \boldsymbol{AC}.y = \boldsymbol{Q}.y \end{matrix}$$

$$\Longleftrightarrow \begin{bmatrix} \boldsymbol{AB}.x & \boldsymbol{AC}.x \\ \boldsymbol{AB}.y & \boldsymbol{AC}.y \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}.x \\ \boldsymbol{Q}.y \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \boldsymbol{AB}.x & \boldsymbol{AC}.x \\ \boldsymbol{AB}.y & \boldsymbol{AC}.y \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{Q}.x \\ \boldsymbol{Q}.y \end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{det} \begin{bmatrix} \boldsymbol{AC}.y & -\boldsymbol{AC}.x \\ -\boldsymbol{AB}.y & \boldsymbol{AB}.x \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}.x \\ \boldsymbol{Q}.y \end{bmatrix}$$

$$det = \boldsymbol{AB}.x \times \boldsymbol{AC}.y - \boldsymbol{AC}.x \times \boldsymbol{AB}.y$$

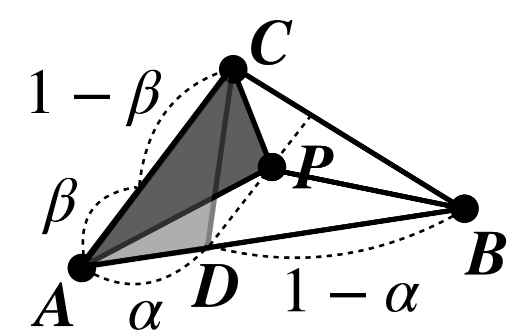
$$\Longleftrightarrow \begin{matrix} \alpha = (\boldsymbol{AC}.y \times \boldsymbol{Q}.x - \boldsymbol{AC}.x \times \boldsymbol{Q}.y)/det \\ \beta = (\boldsymbol{AB}.x \times \boldsymbol{Q}.y - \boldsymbol{AB}.y \times \boldsymbol{Q}.x)/det \end{matrix}$$

It is inside in the triangle iff:

$$\beta \geq 0 \quad \& \quad \beta \leq 1 \quad \& \; \alpha \geq 0 \quad \& \quad \alpha + \beta \leq 1$$

Its barycentric coordinates:

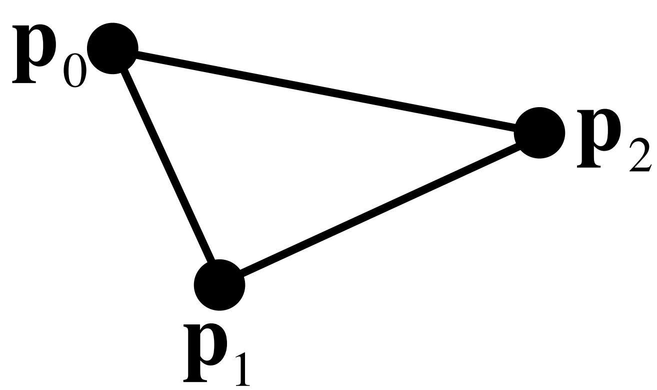
$$w_A = 1 - \alpha - \beta$$
$$w_B = \alpha$$
$$w_C = \beta$$



$$w_B = \frac{\Delta \boldsymbol{APC}}{\Delta \boldsymbol{ABC}} = \frac{\Delta \boldsymbol{ADC}}{\Delta \boldsymbol{ABC}} = \alpha$$

Quiz

- Where is $\mathbf{p}$? And, explain the **barycentric coordinates**.



$$\mathbf{p} = \frac{1}{6}\mathbf{p}_0 + \frac{1}{3}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_2$$

The End